Inequalities

Find all x s.t. $x+5 \geq 2x+1$

$\Leftrightarrow x+4 \geq 2x$

$\Leftrightarrow 4 \geq x$

$x^2 \geq 0$ if $x \in \mathbb{R}$

$(a+b)^2 \geq 4ab$ if $a,b \in \mathbb{R}$

$\Leftrightarrow a^2+b^2+2ab \geq 4ab$

$\Leftrightarrow a^2-2ab+b^2 \geq 0$

$\Leftrightarrow (a-b)^2 \geq 0$

$a,b \geq 0$

$(a+b)^2 \geq 4ab$ $\quad \frac{(a+b)^2}{4} \geq ab$

$\Rightarrow \frac{a+b}{2} \geq \sqrt{ab}$ $\quad (a \geq 0, b \geq 0)$

arithmetic-geometric inequality

$\sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2} \geq \sqrt{ab} \underset{(*)}{\geq} \frac{2}{\frac{1}{a}+\frac{1}{b}}$, if $a>0$, $b>0$

QM AM GM HM

$(*) \Leftrightarrow \frac{1}{a}+\frac{1}{b} \geq \frac{2}{\sqrt{ab}} \Leftrightarrow$

$\frac{1}{a} - \frac{2}{\sqrt{ab}} + \frac{1}{b} \geq 0 \Leftrightarrow \left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}\right)^2 \geq 0$

Applications: $a,b,c \in \mathbb{R}^+$

$$a^2+b^2+c^2 \geq ab+bc+ca$$

AM-GM

$x=a^2,\ y=b^2$

$$\frac{x+y}{2} \geq \sqrt{xy}$$

$$\frac{a^2+b^2}{2} \overset{?}{\geq} ab$$

$$\frac{b^2+c^2}{2} \geq bc$$

$$\frac{c^2+a^2}{2} \geq ca$$

Cauchy-Schwarz inequality

$$\sqrt{(a_1^2+a_2^2+\ldots+a_n^2)(b_1^2+b_2^2+\ldots+b_n^2)} \geq |a_1b_1+\ldots+a_nb_n| \quad \left(\frac{b_1}{a_1}=\frac{b_2}{a_2}=\ldots=\frac{b_n}{a_n}=\lambda\right)$$

$n=2$:

$$\sqrt{a^2+b^2}\sqrt{c^2+d^2} \geq |ac+bd|$$

Application:

$$a^2+b^2+c^2 \geq ab+bc+ca$$

$$\Leftrightarrow \sqrt{a^2+b^2+c^2}\sqrt{b^2+c^2+a^2} \geq ab+bc+ca$$

Proof of CS:

$x \in \mathbb{R}$

$$\underbrace{(a_1 - xb_1)^2 + (a_2 - xb_2)^2 + \ldots + (a_n - xb_n)^2}_{} \geq 0$$

$$= \underbrace{(a_1^2 + \ldots + a_n^2)}_{A} + x^2 \underbrace{(b_1^2 + \ldots + b_n^2)}_{B} - 2x \underbrace{(a_1 b_1 + \ldots + a_n b_n)}_{C}$$

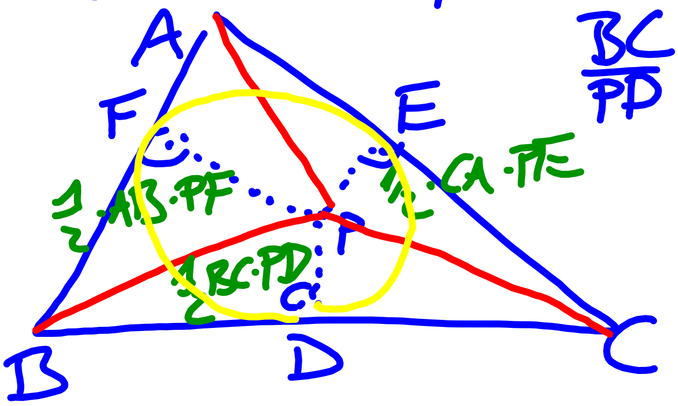
$$Bx^2 \underbrace{+ \cancel{\ldots}}_{-2C} x + \underbrace{\cancel{\ldots}}_{A} \geq 0 \quad \forall x \in \mathbb{R}$$

quadratic with at most 1 zero

$$\Rightarrow (-2C)^2 - 4AB \leq 0$$

$$x_{1,2} = \frac{2C \pm \sqrt{(-2C)^2 - 4AB}}{2B} \quad \Rightarrow C^2 \leq AB$$

Questionnaire, Pb. 10



$$\frac{BC}{PD}+\frac{CA}{PE}+\frac{AB}{PF} \geq ?$$

$$BC\cdot PD + CA\cdot PE + AB\cdot PF = 2\cdot \Delta$$

area of ABC

$$\left(\frac{BC}{PD}+\frac{CA}{PE}+\frac{AB}{PF}\right)\underbrace{\left(BC\cdot PD+CA\cdot PE+AB\cdot PF\right)}_{=2\Delta}$$

$$=\left(\left(\sqrt{\frac{BC}{PD}}\right)^2+ \ldots\right)\left(\left(\sqrt{BC\cdot PD}\right)^2+\ldots\right)$$

$$\geq (BC+CA+AB)$$

$$\frac{BC}{PD}+\frac{CA}{PE}+\frac{AB}{PF} \geq \frac{BC+CA+AB}{2\Delta}$$

equality: $\Leftrightarrow \dfrac{\sqrt{BC\cdot PD}}{\sqrt{\frac{BC}{PD}}} = \dfrac{\sqrt{CA\cdot PE}}{\sqrt{\frac{CA}{PE}}} = \ldots$

$$PD^{\cancel{2}} = PE^{\cancel{2}} = PF^{\cancel{2}} \Leftrightarrow P\ \text{incenter}$$